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A Soluble Kinetic Model for Spinodal Decomposition

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We compare the two-dimensional voter model with approximate theories for spinodal decomposition. The cluster size distribution and the short-time dynamics of the voter model are studied by means of a Monte Carlo simulation. The time-dependent structure factor and the long-time scaling of the voter dynamics are known analytically.

KEY WORDS: Voter model in two dimensions; spinodal decomposition.

1. INTRODUCTION

We accidentally came across a striking similarity between what on the surface would appear to be completely different systems. In Fig. 1b we show subsequent instantaneous configurations of a two-dimensional Lennard-Jones fluid from a molecular dynamics simulation by Koch *et al.*[1] The fluid is quenched to low temperatures at coexistence. As time proceeds, the fluid clusters into a stable low-density (gas) and high-density (fluid) phase—the standard setup for spinodal decomposition. In Fig. 1a we show the Monte Carlo simulation of a stochastic Ising model, known as the *voter model*,[2] by Cox and Griffeath.[3] The system is started in a random configuration of spins. For long times it approaches either all spins up or all spins down, each with probability 1/2.

Spinodal decomposition (nucleation, metastability, etc.) has been studied intensively for many years and is by far not yet a closed chapter of physics; see the recent reviews in refs. 4–7. To complement experiment and theory, very long runs on kinetic Ising models have been carried through. Now, the distinguishing feature of the voter model is its solubility. Just as for the one-dimensional Glauber dynamics, the time-dependent correlation

This paper is dedicated to Nico van Kampen on the occasion of his 67th birthday.

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functions can be obtained in closed form. As is demonstrated in Fig. 1a, the voter model displays spinodal decomposition similar to more realistic systems. For us these were reasons enough to give a closer look at the voter model, in particular, to compare it with standard theories of spinodal decomposition.

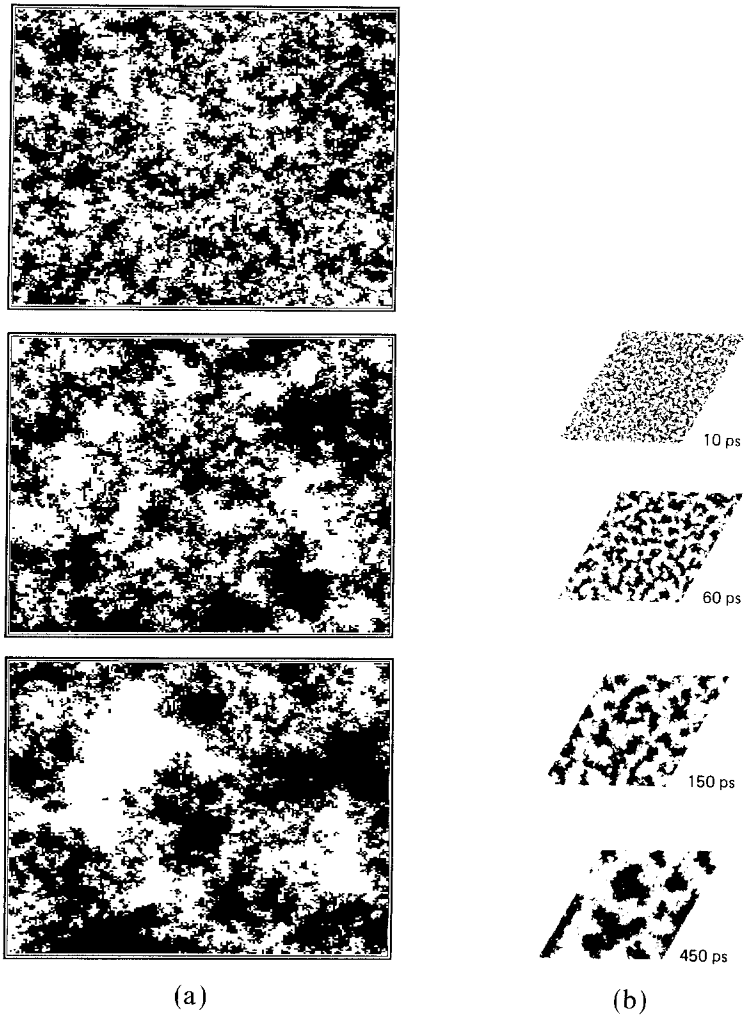


Fig. 1. (a) Instantaneous configurations of the voter model at 50, 450, and 1250 Monte Carlo time steps. Reprinted from Cox and Griffeath.[3] (b) Instantaneous configurations of a two-dimensional, classical Lennard-Jones fluid. The temperature is kept constant. The times are measured in picoseconds corresponding to argon just above the triple point. Reprinted from Koch *et al.*[1]

We explain the voter dynamics and its solution in Section 2 and 3. In Section 4 we review the scaling theory of Cox and Griffeath. Their approach could be of interest also for other systems. Short-time dynamics and cluster size distributions are studied numerically in Section 5.

A mean-field-type approximation to spinodal decomposition is equivalent to the diffusion of a particle in a bistable potential starting at the unstable equilibrium point. Various approximate theories for the approach to the stable equilibria have been proposed. Van Kampen found an exact solution for a particular choice of the bistable potential.[8] This solution serves as a useful check on the theories. In the same spirit we hope that the voter model will provide further insight into the complicated kinetics of spinodal decomposition.

2. THE VOTER MODEL

We consider spins on a simple hypercubic lattice with lattice constant a. (There is no difficulty in extending the theory to other lattices.) Distances are measured in units of a, which amounts to setting $a = 1$. The spin at site x, $x \in Z^d$, takes values ± 1, i.e., $\sigma(x) = \pm 1$. A spin configuration is denoted by $\sigma = \{\sigma(x) \mid x \in Z^d\}$. The voter dynamics is given by the following rule: if at a given site x the spin $\sigma(x) = 1$ (-1) and if all neighboring spins are up (down), then $\sigma(x)$ does not flip; if one of the neighboring spins is down (up), then $\sigma(x)$ flips to -1 $(+1)$ with rate $\lambda/2d$; if two neighboring spins are down (up), $\sigma(x)$ flips with rate $2\lambda/2d$; etc. The name "voter" comes from interpreting $+$ as yes and $-$ as no. The current opinion changes then according to the opinions of nearby friends.

Let $c_x(\sigma)$ be the rate for the spin at x to flip when the spin configuration is σ. Then, in general, the (backward) master equation of the stochastic dynamics reads

$$\frac{d}{dt} f_t(\sigma) = L f_t(\sigma) \tag{2.1}$$

with generator

$$Lf(\sigma) = \sum_x c_x(\sigma)[f(\sigma^x) - f(\sigma)] \tag{2.2}$$

Here σ^x denotes the spin configuration σ with the spin at site x flipped, i.e., $\sigma^x(x) = -\sigma(x)$ and $\sigma^x(y) = \sigma(y)$ for $y \neq x$. The formal solution to (2.1) reads

$$f_t(\sigma) = e^{Lt} f(\sigma) = \sum_{\sigma'} e^{Lt}(\sigma, \sigma') f(\sigma') \tag{2.3}$$

and defines the probability of the configuration σ' at time t given that the initial configuration is σ. Here $e^{Lt}(\sigma, \sigma')$ is the transition probability.

The voter dynamics is a particular case of a general class of exactly soluble models constructed in such a way that L applied to some function f does not increase its degree. Let us first consider just linear functions, such as $f(\sigma)=\sigma(z)$. Clearly, for Lf to be linear, too, the flip rates have to be of the form

$$c_x(\sigma)=a(x)+\sigma(x)\left[\sum_{y\neq x} g(x, y)\,\sigma(y)\right] \tag{2.4}$$

Here $a(x)$ and $g(x, y)$ are as yet unspecified coefficients. Physically, we would impose translation and rotation invariance. We also require spin-flip symmetry. Then, in the case of *short range* interactions, the flip rates become

$$c_x(\sigma)=\lambda\left[1-\gamma\frac{1}{2d}\sum_{y,|x-y|=1}\sigma(x)\,\sigma(y)\right] \tag{2.5}$$

$|\gamma|\leqslant 1$. The λ sets the overall time scale. The significance of γ will become clear in a moment. For $\gamma=1$, (2.5) are the flip rates of the voter model.

Physically, an important general constraint on the flip rates is that they should satisfy detailed balance. This means that, in equilibrium, a given history of spin flips and the time-reversed history have the same probability. This property is, so to speak, inherited from the true microscopic dynamics. Now, for dimension $d=1$ the flip rates $c_x(\sigma)$ of (2.5) are identical to the one of Glauber[9] with $\gamma=\tanh\beta$, where β the is inverse temperature. Therefore, the equilibrium state is the nearest-neighbor Ising model and detailed balance holds. Unfortunately, for $d\geqslant 2$, there is no way to satisfy detailed balance. In this sense the flip rates (2.5) are unphysical. Why bother, then? Our philosophy here is to trade detailed balance against exact solubility. We believe that the voter model, although lacking detailed balance, still provides some insight into the mechanism of spinodal decomposition.

The exact solubility of the voter model can be seen most directly by considering the hierarchy of time-dependent correlation functions. Let us work out the first two equations in the hierarchy,

$$\frac{d}{dt}\langle\sigma(x)\rangle_t=2\lambda\left\{\gamma\frac{1}{2d}\sum_{e,|e|=1}\left[\langle\sigma(x+e)\rangle_t-\langle\sigma(x)\rangle_t\right]\right\} \tag{2.6}$$

$$\begin{aligned}\frac{d}{dt}\langle\sigma(x)\,\sigma(y)\rangle_t=2\lambda\Bigg\{\gamma\frac{1}{2d}\sum_{e,|e|=1}\big[&\langle\sigma(x+e)\,\sigma(y)\rangle_t\\&+\langle\sigma(x)\,\sigma(y+e)\rangle_t-2\langle\sigma(x)\,\sigma(y)\rangle_t\big]\Bigg\}\end{aligned} \tag{2.7}$$

with $\langle\sigma(x)\,\sigma(x)\rangle_t=1$. The average, $\langle\cdot\rangle_t$, is with respect to the distribution of spins at time t. Clearly, the hierarchy of correlations decouples.

For Hamiltonian dynamics a decoupling of the BBGKY hierarchy is equivalent to independent particle motion. This property does not hold for stochastic dynamics, however, as can be seen from the second equation. Let us think of x, y as positions of "particles." Then, if they are apart, they move independently according to the dynamics given by the first equation. If the particles are next to each other, then, because of $\langle\sigma(x)\,\sigma(x)\rangle_t=1$, some terms in (2.7) degenerate, which corresponds to an interaction. On the level of the correlation functions this interaction reflects that we are not just studying the motion of independent spins.

Equations (2.6), (2.7) generalize to n-point functions. Rather than write down the appropriate linear equation, it is more instructive to give, in a way, its solution. We first discuss the case $\gamma=1$. We want to compute $\langle\prod_{j=1}^n\sigma(x_j)\rangle_t$, where all x_j are distinct. For this purpose we consider n random walks $x_j(t)$, $j=1,\ldots,n$, on the lattice Z^d. They start at x_j, i.e., $x_j(0)=x_j$, $j=1,2,\ldots,n$. The random walkers jump with rate λ/d to nearest neighbor lattice sites. The random walkers are not independent. Rather, when jumping on top they annihilate each other, i.e., both walkers disappear. At time t only the walkers with label $j\in A(t)\subset\{1,\ldots,n\}$ survive. Note that $A(t)$ is a random set. $A(t)$ empty corresponds to no walker present at time t. The n-point correlation function is then given by

$$\left\langle\prod_{j=1}^n\sigma(x_j)\right\rangle_t=E\left(\left\langle\prod_{j\in A(t)}\sigma(x_j(t))\right\rangle\right)\tag{2.8}$$

On the right hand side, $\langle\cdot\rangle$ refers to the average in the initial (time $t=0$) state and E is the average over all annihilating random walks.

For $\gamma<1$, (2.8) has to be changed in such a way that each walker jumps with rate $\lambda\gamma/d$ and the path of each walker is weighted with the exponential $\exp[-t2\lambda(1-\gamma)]$, where t is the time of the walk.

Equation (2.8) gives a handle on the steady states of the voter model. We start the system in a homogeneous state, where spins are independent and have an average magnetization m. Then (2.8) simplifies to

$$\left\langle\prod_{j=1}^n\sigma(x_j)\right\rangle_t=E(m^{|A(t)|})\tag{2.8$'$}$$

where $|A(t)|$ is the number of elements in $A(t)$.

Let us again first consider the case $\gamma=1$. Since, if all neighbors agree, the flip rate is zero, once the system has formed a large cluster, it may resolve only through flips at the boundary. For dimensions $d=1$ and 2 the growth dominates: As $t\to\infty$ in the steady state, all spins are up with probability $\frac{1}{2}(1+m)$ and down with probability $\frac{1}{2}(1-m)$. The mathematical reason behind this is that two walkers will meet and hence annihilate each other with probability one. Therefore $|A(t)|=0$ for n even and $|A(t)|=1$ for n odd as $t \to \infty$. On the other hand, for $d \geqslant 3$, two walkers have a finite probability to miss each other. The voter model then has a one-parameter family of steady states, labeled by the average magnetization. By (2.8′), the steady-state covariance is

$$\begin{aligned}&\langle [\sigma(x)-m][\sigma(y)-m]\rangle_s \\ &\quad =(1-m^2)\,\mathrm{Prob}\{\text{walkers } x(t) \text{ and } y(t),\ x(0)=x, \\ &\qquad y(0)=y, \text{ will meet at some time}\} \end{aligned} \tag{2.9}$$

Thus, for $d \geqslant 3$ the voter model maintains nontrivial steady states. Opinions do not become unanimous. This, at first sight rather surprising feature, triggered a fairly detailed study of the voter model.[2] In fact, the large-distance behavior in the steady state is that of a Gaussian, massless free field with covariance $1/k^2$ in Fourier space.[10,11]

If $|\gamma|<1$, there is no mechanism to maintain large clusters. There is only one steady state with $m=0$. From (2.8′) the two-point function in the steady state may be read off. We denote by $p(x, t)$ the probability for a single random walker who starts at x to be absorbed at the origin at time t. The nearest neighbor jump rate of the walker is $2\lambda\gamma/d$. Then, in the steady state,

$$\begin{aligned}&\langle \sigma(x)\rangle_s = 0 \\ &\langle \sigma(x)\,\sigma(y)\rangle_s = \int_0^\infty dt\, e^{-t2\lambda(1-\gamma)}\, p(x-y, t)\end{aligned} \tag{2.10}$$

The static structure factor is

$$\sum_x e^{ik.x}\langle \sigma(0)\,\sigma(x)\rangle_s \cong \frac{1}{1+k^2[\gamma/(1-\gamma)d]} \tag{2.11}$$

for small k. Therefore, the correlation length equals $[\gamma/(1-\gamma)d]^{1/2}$ and diverges as $\gamma \to 1_-$.

We conclude that for $d \geqslant 3$ the voter model ($\gamma=1$) is critical. The anomalous dimension is $\eta=0$. A bit of further work shows that the critical dynamical exponent is $z=2$.[12] The critical behavior is mean field. The factor $1-\gamma$ regulates the distance from the "critical point."

However, for dimension $d=2$, the voter model exhibits spinodal decomposition characteristic for "low" temperatures. For long times the system decomposes into large droplets of up spins and large droplets of down spins. The goal of this paper is to understand the decomposition process on a quantitative level.

3. THE TIME-DEPENDENT STRUCTURE FACTOR

We focus now on the voter model in two dimensions. We have already set the lattice constant $a=1$. Similarly, we measure time in units of the inverse flip rate λ^{-1}, which amounts to setting $\lambda=1$. Our expressions then look dimensionally wrong. It is a simple matter, however, to reintroduce λ and a at the appropriate places. First we compute the exact structure factor $S(k, t)$ and compare it with the standard scaling theories for spinodal decomposition.

We assume that initially spins are independent with zero magnetization, $\langle \sigma(x) \rangle = 0$. Then also at time t

$$\langle \sigma(x) \rangle_t = 0 \tag{3.1}$$

The two-point function $\langle \sigma(x)\, \sigma(y) \rangle_t$ depends only on the difference $x-y$. We define the time-dependent structure factor by

$$S(k, t) = \sum_x e^{ikx} S(x, t) = \sum_x e^{ikx} \langle \sigma(x)\, \sigma(0) \rangle_t \tag{3.2}$$

with $k \in [-\pi, \pi]^2$, the first Brillouin zone. In (2.8′), specialized to $n=2$, we have either $A(t) = \{1, 2\}$ or $A(t) = \phi$. In the first case $\langle \sigma(x_1(t))\, \sigma(x_2(t)) \rangle = 0$. Therefore, $\langle \sigma(x)\, \sigma(0) \rangle_t$ equals the probability for two random walkers who start at x and 0 to annihilate each other before time t. We go over to relative distance between walkers. It moves as a single random walker on Z^2 with doubled jump rate $2\lambda/d = \lambda = 1$. The random walk is absorbed when hitting the origin. Therefore

$$\begin{aligned} S(x, t) &= \text{Prob}\{x(0) = x;\, x(s) = 0 \text{ for some time } s,\ 0 \leqslant s \leqslant t\} \\ &= \text{probability of absorption before time } t \end{aligned} \tag{3.3}$$

The absorption probability is not known in closed form.

It is advantageous to rewrite (3.3) in a slightly different form. Let L be the generator for the random walk with ansorption at the origin. Let L_0 be the generator for the free random walk,

$$L_0 f(x) = \sum_{e, |e| = 1} [f(x+e) - f(x)] \tag{3.4}$$

By first-order perturbation

$$e^{Lt} = e^{L_0 t} + \int_0^t ds\, e^{L_0(t-s)}(L - L_0)\, e^{Ls} \tag{3.5}$$

Now, in terms of the generator the absorption probability in (3.3) is

$$1 - \sum_{y \neq 0} e^{Lt}(x, y) \tag{3.6}$$

Combining (3.6) with (3.5), we obtain

$$S(x, t) = e^{L_0 t}(x, 0) + 2 \int_0^t ds\, e^{L_0 s}(x, 0)\, p(t-s) \tag{3.7}$$

with

$$p(t) = \sum_{y \neq 0} e^{Lt}(e, y) \tag{3.8}$$

$|e| = 1$. The x dependence is only in the free walk. We recognize $p(t)$ as the survival probability at time t for a random walk with absorption at the origin starting at a site right next to the origin. For large t,[13]

$$p(t) \cong \pi/\log t \tag{3.9}$$

Therefore, for long times and sufficiently small k the structure function is given by

$$S(k, t) = e^{-tk^2} + 2\pi \int_0^t ds\, e^{-sk^2} \frac{1}{\log(t-s)} \tag{3.10}$$

It is understood that for $(t-s)$ small, the logarithm should be modified. The properties to be discussed are independent of the precise short-time behavior.

For systems with a nonconserved order parameter, such as the Ising antiferromagnet and certain disorder–order transitions, as a sort of empirical evidence the normalized structure factor

$$\tilde{S}(k, t) = S(k, t) \Big/ \int_{\mathrm{BZ}} dk\, k^2 S(k, t) \tag{3.11}$$

is of scaling form,

$$\tilde{S}(k, t) = \kappa_0(t)^{-d}\, F(|k|/\kappa_0(t)) \tag{3.12}$$

at least with a weakly time-dependent scaling function F.[14,15] The maximum of $k^2\tilde{S}(k, t)$ turns out to be a convenient measure of the coarseness of the clustering.[16] One finds that this maximum tends to zero proportional to $\kappa_0(t)$. The Langer–Bar'on–Miller theory predicts a linear growth of $\tilde{S}(0, t)$.[17]

It is amazing that the structure factor of the two-dimensional voter model follows precisely the pattern set forth by phenomenological theories. The characteristic length scale grows as

$$\kappa_0(t)^{-1} \cong \sqrt{t} \tag{3.13}$$

Let us first verify the scaling ansatz. We note that the normalization factor is

$$\int_{\mathrm{BZ}} dk\, k^2 S(k, t) = -\sum_{e, |e|=1} [S(e, t) - S(0, t)] = 4p(t) \cong \frac{4\pi}{\log t} \tag{3.14}$$

Therefore

$$\frac{1}{t}\tilde{S}(k, t) = \frac{\log t}{4\pi t}\, e^{-k^2 t} + \frac{1}{2k^2 t}\int_0^{k^2 t} du\, e^{-u}\, \frac{1}{1 + [\log(1 - u/k^2 t)/\log t]} \tag{3.15}$$

For large t the first term is negligible and the second term, considered as a function of $w^2 = k^2 t$, becomes time independent. Thus we obtain the exact scaling function

$$F(w) = \frac{1}{2w^2}\int_0^{w^2} du\, e^{-u} = \frac{1}{2w^2}(1 - e^{-w^2}) \tag{3.16}$$

In Fig. 2 we compare the scaling function $F(w)$ with the true structure factor for various times. With the exception of short times, scaling is well satisfied.

In Fig. 3 we plot the maximum of $k^2\tilde{S}(k, t)$ as a function of t. It folows the expected $1/\sqrt{t}$ behavior.

Without any approximation

$$\tilde{S}(0, t) = \frac{1}{2p(t)}\int_0^t ds\, p(s) \tag{3.17}$$

Its time derivative is approximately

$$\frac{d}{dt}\tilde{S}(0, t) = \frac{1}{2}\left(1 + \frac{1}{t}\int_0^t ds \log s\right) \tag{3.18}$$

which implies an essentially linear growth of $\tilde{S}(0, t)$.

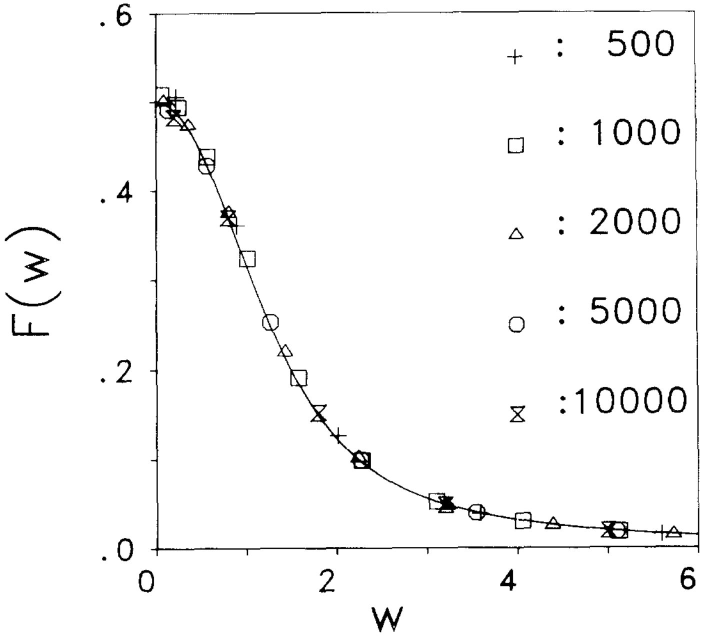


Fig. 2. The scaling function $F(w)$ and the structure factor $\tilde{S}(k, t)/t$ for times $t = (+)$ 500, (□) 1000, (△) 2000, (○) 5000 (⧗) 10,000.

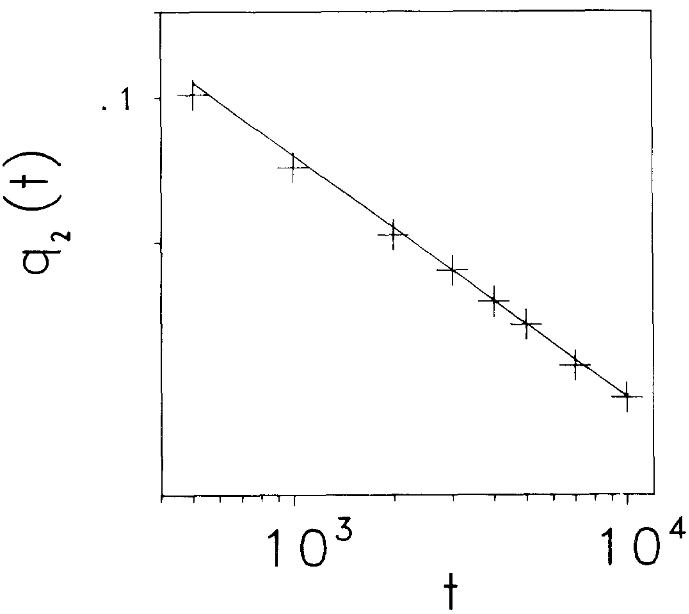


Fig. 3. The maximum of $k^2\tilde{S}(k, t)$ as a function of time t. The slope is -0.5.

4. THE SCALING THEORY OF COX AND GRIFFEATH

In a beautiful piece of work, Cox and Griffeath[3] study the clustering of the two-dimensional voter model for long times. It should be most instructive to analyze physically more realistic models of spinodal decomposition along similar lines.

The basic idea of Cox and Griffeath is to consider block spins, with however, a size which depends on time. Let $\Lambda(\alpha)$ be the square centered at the origin with sides of length $t^{\alpha/2}$, $0<\alpha\leqslant 1$. We define the block spin

$$B_t(\alpha)=\frac{1}{|\Lambda(\alpha)|}\sum_{x\in\Lambda(\alpha)}\sigma_t(x) \tag{4.1}$$

Here $|\Lambda(\alpha)|=t^{\alpha}$ is the number of points in $\Lambda(\alpha)$ and σ_t is the (random) spin configuration at time t. Clearly, $|B_t(\alpha)|\leqslant 1$. We are interested in the distribution of the block spin for large t; in particular, in how it depends on the power α. Let us first discuss the extreme cases: If $\alpha=0$, then the block size is of order one. Therefore, we expect to find either $B_t(0)=1$ or $B_t(0)=-1$ each one of them with probability 1/2. For an initial state with magnetization m the probabilities would be $\frac{1}{2}(1+m)$ and $\frac{1}{2}(1-m)$. On the other hand, if $\alpha\geqslant 1$, i.e., if the side length grows faster than $\sqrt{t}$, then the block typically contains many clusters and $B_t(\alpha)=m$ with probability one as $t\to\infty$. The borderline $\alpha=1$ follows from considering $\langle[B_t(\alpha)-m]^2\rangle$ and using (3.7). So what about the intermediate values of α?

The answer comes out in a surprisingly neat form. Let us define the Fisher–Wright difussion process. It lives on the interval $[-1,1]$ and is governed by the backward generator

$$Lf(w)=\frac{1}{2}(1-w^2)\frac{\partial^2}{\partial w^2}f(w) \tag{4.2}$$

$|w|\leqslant 1$. Diffusions of this type come up in genetics and have been studied extensively.[18,19] Let W_t denote the diffusion process corresponding to (4.2). W_t diffuses in the interval $[-1,1]$ with a spatially dependent diffusion coefficient which vanishes linearly at the boundary. Still, W_t can reach the boundary, where it is absorbed. In fact, W_t reaches the boundary with probability one. Therefore, as $t\to\infty$ the distribution of W_t is $\frac{1}{2}(1+m)\,\delta(w-1)+\frac{1}{2}(1-m)\,\delta(w+1)$ if initially $W_0=m$. The probability distribution at time t is known in terms of Gegenbauer polynomials.[20] A time plot for the symmetric case is given in Fig. 4. There are two time regimes. Initially diffusion spreads out to uniformity. Then the uniform density decreases through absorption at the boundary. In our units the break time is about $\tau=3/4$.

To return to the voter model, Cox and Griffeath prove that, in distribution, the block magnetization has the limit

$$\lim_{t\to\infty} B_t(\alpha) = W_{\log(1/\alpha)} \tag{4.3}$$

Clearly (4.3) reproduces the extreme cases. For $\alpha \to 0$, $\log(1/\alpha) \to \infty$ and the distribution of $B_t(\alpha)$ is concentrated at ± 1, whereas for $\alpha \to 1$, $\log(1/\alpha) \to 0$ and the distribution is concentrated at 0 (in general at m). A typical cluster size is linked roughly with the break time τ. If we set $\tau = 3/4 = \log(1/\alpha)$, then $\alpha = 0.5$ and the characteristic cluster size increases as $\sqrt{t}$, consistent with our results for the structure function.

The convergence in (4.3) is understood in the sense that also the joint distribution of $B_t(\alpha)$, $B_t(\alpha')$ converges to the joint distribution of $W_{\log(1/\alpha)}$, $W_{\log(1/\alpha')}$ (and similarly for higher correlations). Thus, the correlations between various cluster sizes are governed by the Fisher–Wright diffusion, too.

Equation (4.3) shows that the characterization by a single exponent is rather elusive. For large t there are clusters of scales t^α, $0 < \alpha \leqslant 1$. These are weighted, however, by the distribution of $W_{\log}(1/\alpha)$.

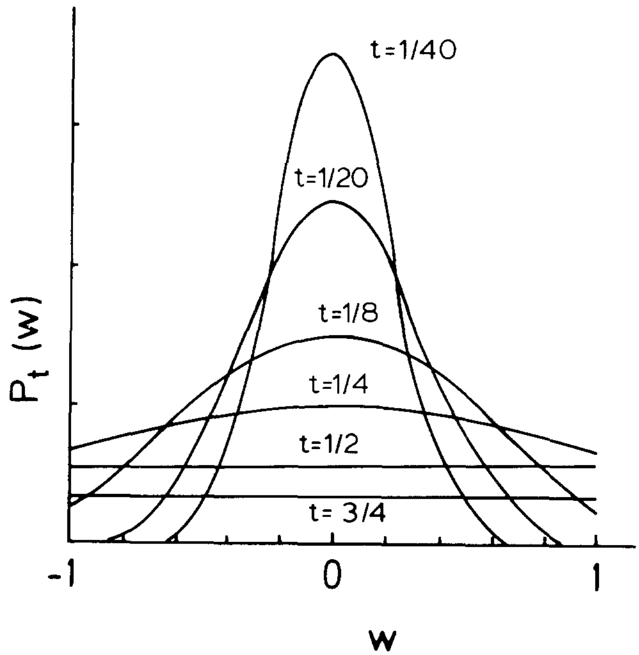


Fig. 4. The probability distribution of the Fisher–Wright diffusion for various times. The initial distribution is $p_0(w) = \delta(w)$.

An obvious criticism to the above construction is that the block magnetization cannot capture the intricate geometric structure of clusters. True enough. But also the Ising model at the critical point has complicated spin clusters. Still, we are able to discuss the critical behavior through simple behavior of measurable quantities. The issue is how fine-grained a detail we need in order to understand and to describe spinodal decomposition.

5. CLUSTER SIZE DISTRIBUTION AND SHORT-TIME DYNAMICS

The two-point function reflects only a sort of average behavior. A more detailed structural information is provided by the number of clusters $n_l(t)$ of size l at time t. To study it, we have to resort to a Monte Carlo simulation of the voter model.

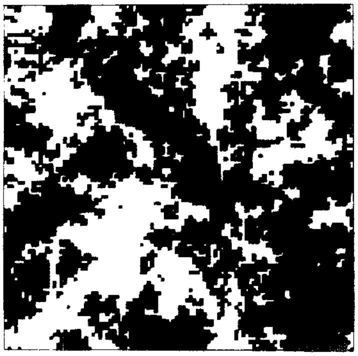
We use a 100×100 square lattice. The starting configuration is random with 50% spin up. The voter dynamics corresponds then to a symmetric quench to low temperatures. The cluster size distribution is determined by an algorithm described in ref. 21.

The half-filled lattice is below the (independent) percolation threshold. Therefore, initially the cluster size distribution decays exponentially,

$$n_l(0) \cong e^{-l/\xi} \tag{5.1}$$

with $\xi \simeq 30$. Under the voter dynamics, in a few flips per spin, the cluster size distribution builds up a power law behavior,

$$n_l(r) \simeq l^{-\tau} \tag{5.2}$$



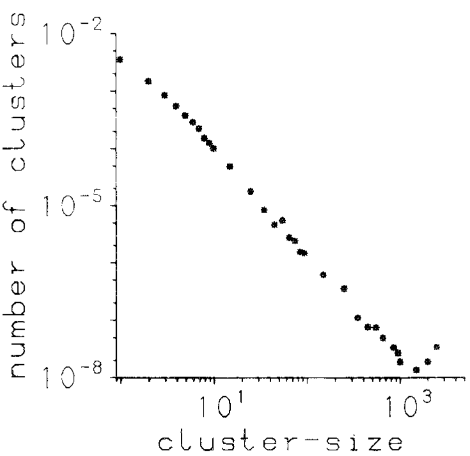


Fig. 5. Cluster size distribution for 200 MCS.

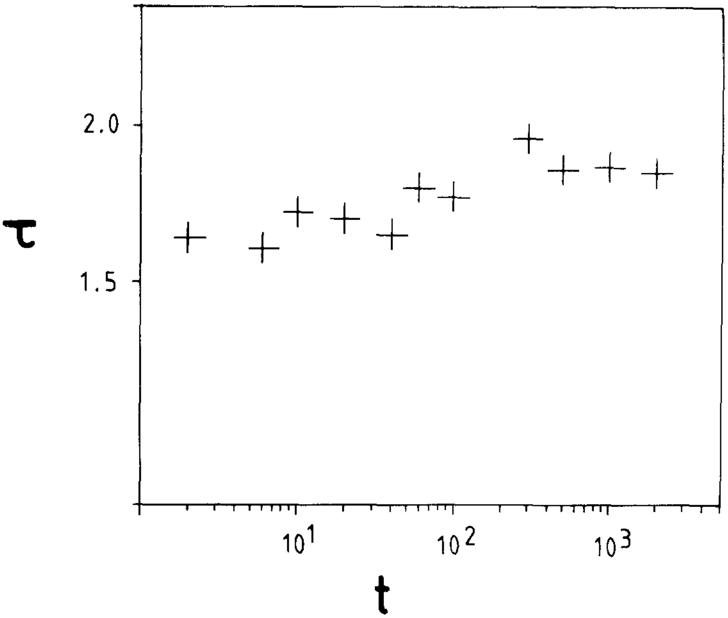


Fig. 6. The exponent τ in (5.2) as a function of time.

for large l (Fig. 5). The exponent τ varies somewhat with time, but does not show any systematic time dependence at least up to 1000 MCS (Monte Carlo time steps) (Fig. 6). We expect that the right-hand side of (5.2) should be supplemented by a time-dependent cutoff function which suppresses the occurrence of clusters of a linear extension larger than some

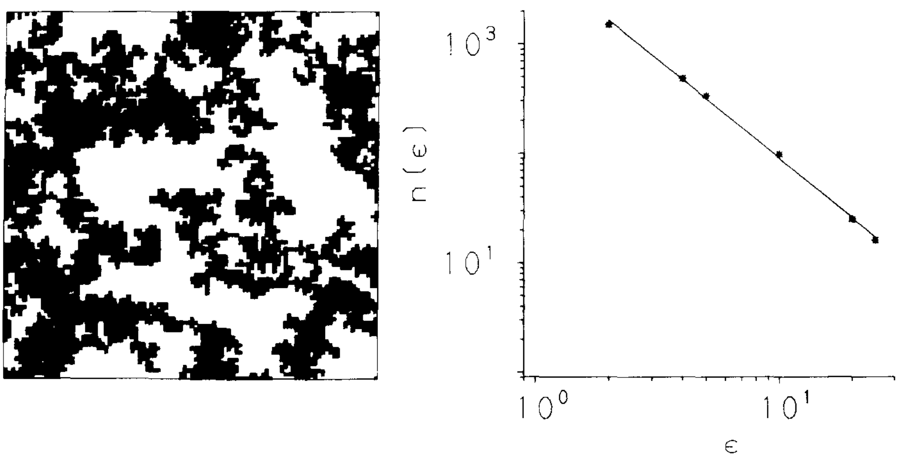


Fig. 7. The largest cluster and its coverage for 60 MCS.

suitable correlation length. No such cutoff could be meaningfully extracted from our data.

Another interesting feature is the almost instantaneous appearance of very large percolating clusters (Fig. 7). These clusters grow until about 20 MCS, after which they break up and continue to grow later on separately. In the early stage of relaxation, the voter model exhibits a transient percolation structure. A similar phenomenon was observed in ref. 22.

Typical percolation clusters have a fractal dimension smaller than the dimension of the underlying space. For the voter dynamics we have measured the fractal dimension of the largest cluster as a function of time. If N_ε denotes the number of squares of area $(100\varepsilon)^2$ needed to cover the cluster, then its fractal dimension (more precisely its capacity dimension) D is defined by

$$N(\varepsilon) \simeq \varepsilon^{-D} \tag{5.3}$$

for $\varepsilon \ll 1$. In the early stage $D \simeq 1.7$ and D tends to 2 for longer times (Fig. 8). Desai and Denton[23] also observed very large fractal clusters in their molecular dynamics simulation of a two-dimension Lennard-Jones fluid. They determined a fractal dimension of 1.5 at the early stage of the spinodal decomposition.

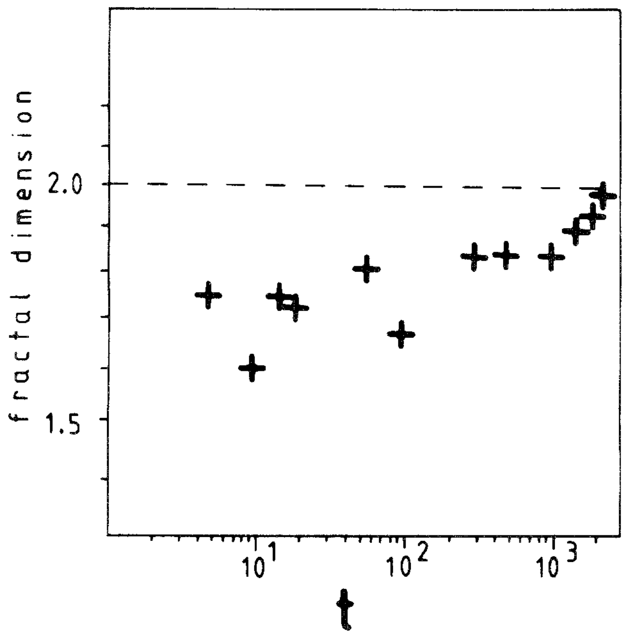


Fig. 8. The fractal dimension of the largest cluster as a function of time.

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